The group of absolutely continuous homeomorphisms of $[0, 1]$ is topologically 2-generated

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The **topological rank** (resp. **generic rank**) of G, denoted by $\operatorname{trk}(G)$ (resp. $\operatorname{grk}(G)$), is the least n for which G is topologically n-generated (resp. generically n-generated).

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- Ω_n is a G_δ set in G^n. Thus, G is generically n-generated iff Ω_n is dense in G^n.
- If $\phi : G_1 \to G_2$ is a continuous group homomorphism with dense image, then $\operatorname{trk}(G_2) \leq \operatorname{trk}(G_1)$.

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S_∞ is topologically 2-generated, as are many other automorphism groups of countable structures. However, a non-archimedean group can never be generically n-generated for any finite n.

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$D^1_+(I)$ is topologically 10-generated. (The actual value of $\mathrm{trk}\left(D^1_+\right)$ is likely lower, but it must be at least 3.)

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A function $f : I \to \mathbb{R}$ is **absolutely continuous** if for every $\epsilon > 0$, there is a $\delta > 0$ such that for every finite, pairwise disjoint collection $((a_i, b_i))_{i<n}$ of open intervals in I, we have

$$\sum_{i<n} b_i - a_i < \delta \implies \sum_{i<n} |f(b_i) - f(a_i)| < \epsilon.$$

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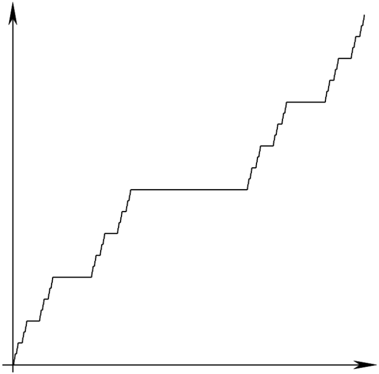
Every Lipschitz continuous function is absolutely continuous, and every absolutely continuous function has bounded variation.

Absolute continuity

Figure: The Cantor staircase is the canonical example of a non-abs cts function.



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(i) f *is absolutely continuous;*

(ii) f *is differentiable almost everywhere,* $f' \in L_1$*, and we have* $f(x) = f(0) + \int_0^x f'(t)\,\mathrm{d}t$ *for all* $x \in I$*;*

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(ii) *f is differentiable almost everywhere, $f' \in L_1$, and we have $f(x) = f(0) + \int_0^x f'(t)\,\mathrm{d}t$ for all $x \in I$;*

(iii) *There exists a map $g \in L_1$ such that $f(x) = f(0) + \int_0^x g(t)\,\mathrm{d}t$ for all $x \in I$.*

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Theorem (Solecki, 1995)

The metric d_{AC} induces a Polish topology on H_+^{AC}, which is finer than the one inherited from H_+.

Aside: subgroups of H_+

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Suffices to show Ω_2 is dense. Fix $f, g \in H_+$, and $\epsilon > 0$. We will build $\tilde{f}, \tilde{g} \in H_+$ such that $d\left(f, \tilde{f}\right) < \epsilon$, $d(g, \tilde{g}) < \epsilon$, and $\Gamma := \left\langle \tilde{f}, \tilde{g} \right\rangle$ is dense in H_+.

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The set $\{(f, g) : \mathrm{Fix}(f) \cap \mathrm{Fix}(g) = \{0, 1\}\}$ is dense, so without loss of generality, we assume f and g do not share any fixed points in $(0, 1)$.

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- Let $\tilde{f}$ have the following properties:
  - $\tilde{f}$ agrees with f on $[\alpha, 1]$
  - $\tilde{f}$ shares no fixed point with $\tilde{g}$ on (y_0, α)
  - $\tilde{f}(x) > x$ for all $x \in (x_0, y_0]$
  - On $[x_{n+1}, x_n]$, $\tilde{f}$ agrees with $\tilde{g}^{-n} \circ \phi_0 \circ \tilde{g}^n$ for n even and $\tilde{g}^{-n} \circ \phi_1 \circ \tilde{g}^n$ for n odd.

Picture

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- $\tilde{f}$ and $\tilde{g}$ do not share any fixed points. Thus, for any $x > 0$ and $y < 1$, there is $h \in \Gamma$ such that $h(x) > y$.
- Using this, one shows that for any $\lambda > 0$, there is $\Phi \in \Gamma$ and some $[a, b] \subseteq I$ such that $a < \lambda < 1 - \lambda < b$, and $\Phi\tilde{f}\Phi^{-1}\restriction_{[a,b]}$ and $\Phi\tilde{g}\tilde{f}\tilde{g}^{-1}\Phi^{-1}$ generate a dense subgroup of $H_+([a, b])$.

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- Need to choose α even smaller to guarantee $d_{AC}(g, \tilde{g}) < \epsilon$.
- Need more ϕ_i's to generate a dense subgroup of $H_+^{AC}([x_1, x_0])$.
- Need to show why a dense subgroup of $H_+^{AC}([a, b])$ can approximate $H_+^{AC}(I)$.